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## Non-additive fusion, Hubbard models and non-locality

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**Abstract.** Within the framework of quantum groups and additive  $R$ -matrices, the fusion procedure allows one to construct higher-dimensional solutions of the Yang–Baxter equation. These solutions lead to integrable one-dimensional spin-chain Hamiltonians. Here fusion is shown to generalize naturally to non-additive  $R$ -matrices, which therefore do not have a quantum group symmetry. This method is then applied to the generalized Hubbard models. Although the resulting integrable models are not as simple as the starting ones, the general structure is that of two spin- $(s \times s')$   $sl(2)$  models coupled at the free-fermion point. An important issue is the probable lack of regular points which give local Hamiltonians. This problem is related to the existence of second-order zeros in the unitarity equation, and arises for the  $XX$  models of higher spins, the building blocks of the Hubbard models. A possible connection between some Lax operators  $L$  and  $R$ -matrices is noted.

### 1. Introduction

The construction and diagonalization of integrable one-dimensional spin-chain Hamiltonians within the framework of the quantum inverse scattering method (QISM) is well known [1–4]. A given integrable model and all its conserved quantities are encoded in an  $R$ -matrix which satisfies the Yang–Baxter equation (YBE). The quantum group approach [5, 6] provides a systematic way to obtain a large class of solutions based on representations of some underlying Lie algebra [7] or super-algebra [8]. By construction, such solutions possess the additivity property which means that the initial double spectral parameter dependence reduces to the difference of the two parameters. The most famous example is the spin- $\frac{1}{2}$   $XXZ$  chain.

On the other hand, models having non-additive  $R$ -matrices have been known to exist for a long time. Examples of such models include Shastry’s solution for the Hubbard model [9, 10], the Chiral Potts models [11–13] and more recently the Bariev models [14–16]. The Hubbard model both in its bosonic and fermionic guises was also generalized to multi-state versions, while retaining the same algebraic structure. Initial versions were first introduced in [17], studied in [18–20], further generalized in [21] and fermionized in [22]. (See also [23] for another possible fermionization scheme.) All these non-additive solutions of the Yang–Baxter equations are isolated and do not yet fit within a general framework.

A general method for constructing solutions to the Yang–Baxter equation out of a given known solution is the fusion method. It works by multiplying the same matrix by itself a certain number of times, at different values of the spectral parameter, and finally multiplying by a projector. This works much the same way as building higher-dimensional representations from tensor products of a smaller one and a final projection on a subspace. For instance, an  $sl(2)$  spin- $s$  solution can be obtained by successive fusions of the spin- $\frac{1}{2}$  solution [24]. It is,

in fact, possible to fuse an arbitrary product of  $R$ -matrices to obtain solutions of the YBE corresponding to most (but not always all [25]) representations of a given Lie algebra [26, 27]. Within the framework of quantum groups, the direct method for finding  $R$ -matrices with a given Lie algebra symmetry, and corresponding to a given representation, consists of solving linear equations [7]. This method and fusion give the same results.

In [28]† fusion was shown to work for a class of models which retained only some aspects of an  $sl(m)$  quantum group structure. Higher-dimensional solutions were obtained by fusion, where no quantum group symmetry and therefore no direct method existed.

In this work I derive fusion equations for non-additive  $R$ -matrices. The results of section 2 are quite general and require the starting matrix to satisfy only a minimal number of properties. The generalized Hubbard models are shown to satisfy these properties. This allows one to construct higher-spin Hubbard models which appear as two copies of a multi-flavour spin- $(s \times s')$  model coupled at the ‘free-fermion’ point. The coupling does not have the simple structure of the starting models. The resulting integrable models have non-additive, unitary,  $R$ -matrices but appear to lack the usual regularity property, which would allow one to obtain *local* Hamiltonians. The source of this lack of regularity is traced back to the spin- $s$  building blocks. These models, for  $s \geq 1$ , are not regular but still allow for local mutually commuting quantities through a limiting procedure from a generic  $q$  value. It is not clear how to implement this for the higher-spin Hubbard models. Possible applications to the Bariev and Chiral Potts models are mentioned in the conclusion. A possible connection between some Lax matrices and  $R$ -matrices is also proposed.

## 2. Fusion for non-additive $R$ -matrices

Given a non-additive solution of the Yang–Baxter equation, it is possible to obtain new solutions provided one has a projector point. Expressions for four fused matrices are found along with the equations they satisfy. The results of this section are general and hold without reference to any particular model. The notation follows closely that of [28]. Here the word fusion is used in the conventional sense, and not in the sense of [21]. For the additive case, Kulish and Sklyanin had already realized that only two properties were needed for fusion to be possible. The Yang–Baxter equation has to be satisfied and a projector point must exist (p 108 of [1]).

Consider a non-additive solution  $R(\lambda_1, \lambda_2)$  of the Yang–Baxter equation

$$R_{12}(\lambda_1, \lambda_2)R_{13}(\lambda_1, \lambda_3)R_{23}(\lambda_2, \lambda_3) = R_{23}(\lambda_2, \lambda_3)R_{13}(\lambda_1, \lambda_3)R_{12}(\lambda_1, \lambda_2). \quad (1)$$

Additivity means that for a proper choice of parametrization, and after eventual transformations such as a gauge (a special similarity transformation on  $R$ ) [7] or twist (a special similarity transformation on  $\check{R}$ ) [29] transformation, one can write  $R_{12}(\lambda_1, \lambda_2) = R_{12}(\lambda_1 - \lambda_2)$ . Most known solutions of the YBE are additive. This includes, in particular, all solutions corresponding to a quantum group symmetry,  $\mathcal{U}_q(\mathcal{G})$ , where  $\mathcal{G}$  is any Lie algebra or super-algebra [7, 8]. Known non-additive solutions include the class of generalized Hubbard models in their bosonic and fermionic forms [21, 22], the chiral Potts models [13] and the Bariev models [14–16].

To implement fusion it is enough that the solution at hand has a projector point. Thus consider any solution  $R$  of the Yang–Baxter equation (1), which becomes proportional to a

† In equation (17)  $f'(\rho)$  and  $f''(\rho)$  should replace  $f'(0)$  and  $f''(0)$ .

projector at some special values of the spectral parameters pair  $(\lambda_1, \lambda_2)$ . Specifically, define the projector  $\pi^{(1)}$  through

$$R(\lambda + \rho, \lambda) \equiv g(\lambda) \pi^{(1)} \tag{2}$$

and let  $\pi^{(2)} = \mathbb{I} - \pi^{(1)}$  be the orthogonal complementary projector. Both projectors are assumed to be independent of  $\lambda$ , and  $\rho$  is some fixed value characteristic of the  $R$ -matrix at hand. I have then verified that the methods used in [28] can be extended to the non-additive setting. This yields the following results.

Let  $S$  be the matrix which diagonalizes both projectors. Define two fused matrices, for  $i = 1, 2$ , by

$$R_{(12)3}^{(i)}(\lambda, \lambda_3) = S_{12}^{-1} \pi_{12}^{(i)} R_{13}(\lambda, \lambda_3) R_{23}(\lambda - \rho, \lambda_3) \pi_{12}^{(i)} S_{12}. \tag{3}$$

The matrices (3) satisfy a YBE where one space is a tensor product of two spaces:

$$\begin{aligned} R_{(12)3}^{(i)}(\lambda, \lambda_3) R_{(12)4}^{(i)}(\lambda, \lambda_4) R_{34}(\lambda_3, \lambda_4) \\ = R_{34}(\lambda_3, \lambda_4) R_{(12)4}^{(i)}(\lambda, \lambda_4) R_{(12)3}^{(i)}(\lambda, \lambda_3) \quad i = 1, 2. \end{aligned} \tag{4}$$

We have thus obtained new  $R$ -matrices. If  $d_i, i = 1, 2, 3$ , are the dimensions of the spaces 1, 2 and 3, and  $\text{tr}(\pi^{(1)}) = d$ , then after deletion of the vanishing rows and columns,  $R_{(12)3}^{(i)}(\lambda, \lambda_3)$  is a  $dd_3$ -dimensional matrix for  $i = 1$ , and  $(d_1d_2 - d)d_3$ -dimensional for  $i = 2$ .

Note also that there is another possible choice of fused matrices obtained by taking the right-hand side of the YBE at the projector point (see (4) of [28]). There is, however, no essential difference with the foregoing choice.

One can then fuse two matrices  $R_{(12)3}^{(i)}(\lambda, \lambda_3)$  to obtain the matrix  $R_{(12)(34)}^{(i)}(\lambda, \mu)$  defined by

$$R_{(12)(34)}^{(i)}(\lambda, \mu) = S_{34}^{-1} \pi_{34}^{(i)} R_{(12)4}^{(i)}(\lambda, \mu - \rho) R_{(12)3}^{(i)}(\lambda, \mu) \pi_{34}^{(i)} S_{34} \quad i = 1, 2. \tag{5}$$

These matrices have dimensions  $d^2$  for  $i = 1$ , and  $(d_1d_2 - d)^2$  for  $i = 2$ . They satisfy two Yang–Baxter equations ( $i = 1, 2$ ):

$$R_{(12)(34)}^{(i)}(\lambda, \mu) R_{(12)5}^{(i)}(\lambda, \lambda_5) R_{(34)5}^{(i)}(\mu, \lambda_5) = R_{(34)5}^{(i)}(\mu, \lambda_5) R_{(12)5}^{(i)}(\lambda, \lambda_5) R_{(12)(34)}^{(i)}(\lambda, \mu) \tag{6}$$

$$R_{(12)(34)}^{(i)}(\lambda, \mu) R_{(12)(56)}^{(i)}(\lambda, \nu) R_{(34)(56)}^{(i)}(\mu, \nu) = R_{(34)(56)}^{(i)}(\mu, \nu) R_{(12)(56)}^{(i)}(\lambda, \nu) R_{(12)(34)}^{(i)}(\lambda, \mu). \tag{7}$$

Now assume that the original  $R$ -matrix is regular and unitary, i.e.

$$R_{12}(\mu, \mu) = c(\mu) \mathcal{P}_{12} \tag{8}$$

$$R_{12}(\lambda, \mu) R_{21}(\mu, \lambda) = f(\lambda, \mu) \mathbb{I} \tag{9}$$

where  $\mathcal{P}$  is the permutation operator and  $R_{21} \equiv \mathcal{P}_{12} R_{12} \mathcal{P}_{12}$ . The function  $f(\lambda, \mu)$  is then symmetric in its arguments, and  $c(\mu)$  is some complex, generically non-vanishing function. The fused matrices (5) inherit the regularity property (8) provided they are correctly normalized, for the corresponding  $c$ -function not to vanish. This can be achieved by the following normalization. Insert a factor of  $(f(\lambda + \rho, \mu))^{-1}$  in the right-hand side of equation (5). As  $R(\mu + \rho, \mu)$  is a non-trivial projector, unitarity (9) implies that  $f(\mu + \rho, \mu)$  vanishes for all values of  $\mu$ . The normalization just introduced cancels this zero in the numerator and leaves a regular fused matrix. Incidentally, the symmetry of  $f$  implies that  $f(\mu, \mu + \rho)$  also vanishes.

The form (5) can be simplified to a more symmetric one:

$$R_{(12)(34)}^{(i)}(\lambda, \mu) = \frac{1}{f(\lambda + \rho, \mu)} S_{12}^{-1} S_{34}^{-1} \pi_{12}^{(i)} \pi_{34}^{(i)} R_{14}(\lambda, \mu - \rho) R_{24}(\lambda - \rho, \mu - \rho) \\ \times R_{13}(\lambda, \mu) R_{23}(\lambda - \rho, \mu) \pi_{12}^{(i)} \pi_{34}^{(i)} S_{12} S_{34} \quad i = 1, 2 \quad (10)$$

where the normalization has been included.

Let  $\partial_i f(\lambda_1, \lambda_2)$  denote the derivative with respect to the  $i$ th slot ( $i = 1, 2$ ). Taking the limit  $\lambda \rightarrow \mu$  for the normalized matrices I find

$$R_{(12)(34)}^{(1)}(\mu, \mu) = c(\mu)c(\mu - \rho) \frac{\partial_2 f(\mu, \mu - \rho)}{\partial_1 f(\mu + \rho, \mu)} \mathcal{P}_{13} \mathcal{P}_{24} S_{12}^{-1} \pi_{12}^{(1)} S_{12} S_{34}^{-1} \pi_{34}^{(1)} S_{34} \quad (11)$$

$$R_{(12)(34)}^{(2)}(\mu, \mu) = c(\mu)c(\mu - \rho) \frac{\partial_1 f(\mu, \mu - \rho)}{\partial_1 f(\mu + \rho, \mu)} \mathcal{P}_{13} \mathcal{P}_{24} S_{12}^{-1} \pi_{12}^{(2)} S_{12} S_{34}^{-1} \pi_{34}^{(2)} S_{34}. \quad (12)$$

The unitarity property is inherited independently from the normalization:

$$R_{(12)(34)}^{(i)}(\lambda, \mu) R_{(34)(12)}^{(i)}(\mu, \lambda) = \frac{f(\lambda - \rho, \mu) f(\lambda, \mu) f(\lambda - \rho, \mu - \rho) f(\lambda, \mu - \rho)}{f(\lambda + \rho, \mu) f(\mu + \rho, \lambda)} \\ \times S_{12}^{-1} \pi_{12}^{(i)} S_{12} S_{34}^{-1} \pi_{34}^{(i)} S_{34} \quad i = 1, 2 \quad (13)$$

where  $R_{(34)(12)}^{(i)}(\lambda, \mu) = \mathcal{P}_{13} \mathcal{P}_{24} R_{(12)(34)}^{(i)}(\lambda, \mu) \mathcal{P}_{13} \mathcal{P}_{24}$ .

Within the framework of the QISM, the quadratic Hamiltonian density of such integrable hierarchies is the derivative at  $\lambda = \mu$  of the matrix  $\check{R}(\lambda, \mu) = \mathcal{P}R(\lambda, \mu)$ . Taking the limit yields

$$\frac{d}{d\lambda} \check{R}_{(12)(34)}^{(i)}(\lambda, \mu)|_{\lambda=\mu} = -\frac{\partial_1^2 f(\mu + \rho, \mu)}{2\partial_1 f(\mu + \rho, \mu)} \check{R}_{(12)(34)}^{(i)}(\mu, \mu) \\ + \frac{1}{2\partial_1 f(\mu + \rho, \mu)} S_{12}^{-1} S_{34}^{-1} \pi_{12}^{(i)} \pi_{34}^{(i)} \frac{d^2}{d\lambda^2} (R_{32}(\lambda, \mu - \rho) \check{R}_{13}(\lambda, \mu) \\ \times \check{R}_{24}(\lambda - \rho, \mu - \rho) R_{23}(\lambda - \rho, \mu))|_{\lambda=\mu} \pi_{12}^{(i)} \pi_{34}^{(i)} S_{12} S_{34}. \quad (14)$$

The first term is proportional to the identity in the fused spaces and may be dropped. The results of [28] in the additive case can be recovered by setting  $f(\lambda, \mu) \rightarrow f(\lambda - \mu)$ , with  $f$  now an even function. (There is an obvious misprint in formula (17) of [28]:  $f'(0)$  and  $f''(0)$  should be replaced by  $f'(\rho)$  and  $f''(\rho)$ .)

An implicit assumption, which does *not* affect the Yang–Baxter equations, was made when deriving the regularity equations (11) and (12):  $f(\lambda + \rho, \mu)$  was taken to vanish like  $(\lambda - \mu)$  for  $\lambda \rightarrow \mu$ . However, this zero can be of second order. This will be case for the Hubbard models studied in the next section. A zero of any order does not affect the unitarity equation (13) as the numerator and denominator compensate to give a finite non-vanishing result. However, it is necessary to perform a second- or higher-order expansion to find the appropriate expressions for the regularity equations. It does not appear to be possible to prove in general, and with a minimal set of assumptions, that regularity still holds. However, the result is finite. An argument in favour of regularity is the unitarity equation which says that  $\check{R}_{12}(\mu, \mu)$  squares to the identity. However, this turns out not to be enough and specific counter-examples are the multi-flavour spin- $s$  models at  $\gamma = \pi/2$ , for  $s \geq 1$ . Note that the above issues of regularity will arise for all fused matrices with non-simple zeros in the unitarity equations of the starting  $R$ -matrix.

In the case of a second-order zero for  $f(\lambda, \mu)$ , and provided the fused matrix is regular, the Hamiltonian density is given by

$$\begin{aligned} \frac{d}{d\lambda} \check{R}_{(12)(34)}^{(i)}(\lambda, \mu)|_{\lambda=\mu} &= -\frac{\partial_1^3 f(\mu + \rho, \mu)}{3\partial_1^2 f(\mu + \rho, \mu)} \check{R}_{(12)(34)}^{(i)}(\mu, \mu) \\ &+ \frac{1}{3\partial_1^2 f(\mu + \rho, \mu)} S_{12}^{-1} S_{34}^{-1} \pi_{12}^{(i)} \pi_{34}^{(i)} \frac{d^3}{d\lambda^3} (R_{32}(\lambda, \mu - \rho) \check{R}_{13}(\lambda, \mu) \\ &\times \check{R}_{24}(\lambda - \rho, \mu - \rho) R_{23}(\lambda - \rho, \mu))|_{\lambda=\mu} \pi_{12}^{(i)} \pi_{34}^{(i)} S_{12} S_{34}. \end{aligned} \tag{15}$$

Regularity implies that the first term is proportional to the identity and may be dropped.

The projector property is the only additional ingredient needed to construct fused matrices. For the  $R$ -matrices based on Lie algebras, the degeneration of a generically invertible  $R$ -matrix to a projector is expected. However, for non-additive matrices one has to verify in every case whether a projector point exists. Note also that the unitarity property (13) for the fused matrix indicates that it may have its own projector point. This in turn implies that fusion may be continued to another level, or even indefinitely as happens in the quantum group framework.

The above fusing scheme will now be applied to the generalized Hubbard models.

### 3. Hubbard fusion and non-locality

We first recall the construction of the multi-state or multi-flavour Hubbard models in their bosonic form<sup>†</sup>. The connection between the  $L$ - and  $R$ -matrices is clarified. Fusion is implemented. The connection between double zeros in the unitarity equation and the lack of regularity is discussed using specific examples.

#### 3.1. A Hubbard primer

The following ‘free-fermions’ or  $XX$  models are building blocks of the Hubbard models. Let  $n, n_1$  and  $n_2$  be three positive integers such that  $n_1 + n_2 = n$ , and  $A, B$  be two disjoint sets whose union is the set of basis states of  $\mathbb{C}^n$ , with  $\text{card}(A) = n_1$  and  $\text{card}(B) = n_2$ . Let  $E^{\alpha\beta}$  be a square matrix with a one at row  $\alpha$  and column  $\beta$  and zeros otherwise. Define

$$\tilde{P}^{(1)} = \sum_{a \in A} \sum_{\beta \in B} (E^{a\beta} \otimes E^{\beta a} + E^{\beta a} \otimes E^{a\beta}) \tag{16}$$

$$\tilde{P}^{(2)} = \sum_{a, a' \in A} E^{aa'} \otimes E^{a'a} + \sum_{\beta, \beta' \in B} E^{\beta\beta'} \otimes E^{\beta'\beta} \tag{17}$$

$$\tilde{P}^{(3)} = \sum_{a \in A} \sum_{\beta \in B} (xE^{aa} \otimes E^{\beta\beta} + x^{-1}E^{\beta\beta} \otimes E^{aa}). \tag{18}$$

Latin indices always belong to  $A$ , while Greek indices belong to  $B$ . The complex twist parameter  $x$  is arbitrary. The free-fermions  $R$ -matrix

$$R(\lambda) = \tilde{P}^{(1)} + \tilde{P}^{(2)} \cos \lambda + \tilde{P}^{(3)} \sin \lambda \tag{19}$$

satisfies the additive Yang–Baxter equation:

$$R_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda - \mu). \tag{20}$$

The interpretation of the multiple flavours in terms of  $sl(2)$  states was done in [28].

<sup>†</sup> See [21], in section 7  $T_{\alpha\alpha'}$  should replace  $t_{\alpha\alpha'}$ .

Coupling two commuting copies of the foregoing models gives the Hubbard models. Where made explicit, the two copies are denoted by unprimed and primed quantities. Let us stress that the copies need *not* be of the same type. For instance, the ‘left’ copy can be  $(n_1, n_2)$ , while the ‘right’ copy is  $(n'_1, n'_2)$  with  $n$  not necessarily equal to  $n'$ . ( $n_i = n'_i = 1$  correspond to the original Hubbard model.) The twist parameters may also differ. One then defines a multi-flavour version of  $\sigma^z$ , the conjugation matrix

$$C = \sum_{\beta \in B} E^{\beta\beta} - \sum_{a \in A} E^{aa} \quad (21)$$

and a diagonal coupling matrix

$$I_{00'}(h) = \cosh(\frac{1}{2}h) \mathbb{I} + \sinh(\frac{1}{2}h) C_0 C'_0 = \exp(\frac{1}{2}h C_0 C'_0). \quad (22)$$

The parameter  $h$  is related to the spectral parameter  $\lambda$  by

$$\sinh(2h) = U \sin(2\lambda) \quad (23)$$

where  $U$  is the coupling constant. One chooses for  $h(\lambda)$  the principal branch which vanishes for vanishing  $\lambda$  or  $U$ . The Lax operator at site  $i$  is equal to

$$L_{0i}(\lambda) = I_{00'}(h) R_{0i}(\lambda) R_{0i'}(\lambda) I_{00'}(h). \quad (24)$$

Their commutation relations for different spectral parameters at a given site are given by

$$R(\lambda_1, \lambda_2) \overset{1}{L}(\lambda_1) \overset{2}{L}(\lambda_2) = \overset{2}{L}(\lambda_2) \overset{1}{L}(\lambda_1) R(\lambda_1, \lambda_2) \quad (25)$$

where  $\overset{1}{L}(\lambda_1) = L(\lambda_1) \otimes \mathbb{I}$ ,  $\overset{2}{L}(\lambda_2) = \mathbb{I} \otimes L(\lambda_2)$ , and

$$R(\lambda_1, \lambda_2) = I_{12}(h_1) I_{34}(h_2) \left[ R_{13}(\lambda_1 - \lambda_2) R_{24}(\lambda_1 - \lambda_2) + \frac{\sin(\lambda_1 - \lambda_2)}{\sin(\lambda_1 + \lambda_2)} \right. \\ \left. \times \tanh(h_1 + h_2) R_{13}(\lambda_1 + \lambda_2) C_1 R_{24}(\lambda_1 + \lambda_2) C_2 \right] I_{12}(-h_1) I_{34}(-h_2). \quad (26)$$

This matrix is non-additive as it is not possible to reduce its spectral parameter dependence to  $\lambda_1 - \lambda_2$ . It satisfies the regularity property

$$R(\lambda_1, \lambda_1) = \mathcal{P}_{13} \mathcal{P}_{24} \quad (27)$$

and the unitarity property

$$R_{12}(\lambda_1, \lambda_2) R_{21}(\lambda_2, \lambda_1) = \cos^2(\lambda_1 - \lambda_2) (\cos^2(\lambda_1 - \lambda_2) - \cos^2(\lambda_1 + \lambda_2) \tanh^2(h_1 - h_2)) \mathbb{I}. \quad (28)$$

The matrix (26) satisfies the Yang–Baxter equation (1), with  $\lambda_i$  and  $h_i$  related through (23).

The Hubbard models have a Lax matrix  $L$  which differs from the intertwiner  $R$ . One may wonder which matrix should be a candidate for fusing, and what is the role of the  $RLL$  relation as opposed to the  $RRR$  one. The matrix  $R$  satisfies the symmetric equation (1), while  $L$  satisfies the asymmetric equation (25). This already singles out the former as the natural object to fuse. Another compelling reason is that  $L$  and (25) are just special asymmetrical limits of  $R$  and (1). Indeed, one easily finds that

$$R_{1234}(\lambda_1, 0) = \frac{1}{\cosh h_1} I_{12}(h_1) R_{13}(\lambda_1) R_{24}(\lambda_1) I_{12}(h_1) = \frac{1}{\cosh h_1} L_{(12)(34)}(\lambda_1). \quad (29)$$

Setting  $\lambda_3 = 0$  in (1) gives (25). One also has

$$R_{1234}(0, \lambda_2) = \frac{1}{\cosh h_2} I_{34}(-h_2) R_{13}(-\lambda_2) R_{24}(-\lambda_2) I_{34}(-h_2) \quad (30)$$

where now the coupling of the two copies is made on the quantum spaces rather than the auxiliary spaces. Setting  $\lambda_1 = 0$  in (1) gives an *RLL* relation. The corresponding quadratic Hamiltonian and all the other conserved quantities are, however, essentially the same as for the auxiliary-space coupling case. Now the *R*-matrix can be seen as an *L*-matrix with couplings on both auxiliary and quantum spaces. The price of this symmetrization is the linear combination in (26) and the loss of additivity. The quadratic Hamiltonian density obtained from *R* at the arbitrary regular point  $\lambda = \mu$  is given by

$$\begin{aligned} \frac{d}{d\lambda} \check{R}(\lambda, \mu)|_{\lambda=\mu} &= \frac{U \cos 2\mu}{2 \cosh 2h} (C_3 C_4 - C_1 C_2) \\ &+ \mathcal{P}_{13} \tilde{P}_{13}^{(3)} (\cosh^2 h - C_2 C_4 \sinh^2 h) + \mathcal{P}_{24} \tilde{P}_{24}^{(3)} (\cosh^2 h - C_1 C_3 \sinh^2 h) \\ &- \frac{1}{2} \sinh(2h) (p_{13}^{(3)} (C_2 + C_4) + p_{24}^{(3)} (C_1 + C_3)) \\ &+ \frac{U}{\cosh 2h} (-\sin(2\mu) \sinh(2h) (\mathcal{P}_{13} \tilde{P}_{13}^{(1)} \mathcal{P}_{24} \tilde{P}_{24}^{(3)} + \mathcal{P}_{13} \tilde{P}_{13}^{(3)} \mathcal{P}_{24} \tilde{P}_{24}^{(1)}) \\ &+ 2 \sin 2\mu \sinh^2 h (p_{13}^{(1)} p_{24}^{(3)} + p_{13}^{(3)} p_{24}^{(1)}) \\ &+ (p_{13}^{(1)} + \cos(2\mu) p_{13}^{(2)} + \sin(2\mu) p_{13}^{(3)}) (p_{24}^{(1)} + \cos(2\mu) p_{24}^{(2)} + \sin(2\mu) p_{24}^{(3)}) \end{aligned} \tag{31}$$

where  $h = h(\mu)$  is given by (23) and  $p_{jk}^{(i)} \equiv \mathcal{P}_{jk} \tilde{P}_{jk}^{(i)} C_j$ ,  $i = 1, 2, 3$  and  $j, k = 1, \dots, 4$ . The indices are interpreted as follows: 1  $\rightarrow$  site-*m*-unprimed-copy, 2  $\rightarrow$  site-*m*-primed-copy, 3  $\rightarrow$  site- $(m + 1)$ -unprimed-copy, 4  $\rightarrow$  site- $(m + 1)$ -primed-copy. It is only at  $\mu = 0$  where this expression reduces to the familiar generalized (bosonic) form of the Hubbard Hamiltonians.

The *I* factors in (26) combine into a similarity transformation. It is, in fact, a special type of gauge transformation, and an equivalent *R*-matrix is given by [10]

$$\begin{aligned} r(\lambda_1, \lambda_2) &= R_{13}(\lambda_1 - \lambda_2) R_{24}(\lambda_1 - \lambda_2) \\ &+ \frac{\sin(\lambda_1 - \lambda_2)}{\sin(\lambda_1 + \lambda_2)} \tanh(h_1 + h_2) R_{13}(\lambda_1 + \lambda_2) C_1 R_{24}(\lambda_1 + \lambda_2) C_2. \end{aligned} \tag{32}$$

The corresponding Lax matrix *l*( $\lambda$ ) is given by

$$r(\lambda, 0) = \frac{1}{\cosh h} l(\lambda) = \frac{1}{\cosh h} R_{13}(\lambda) R_{24}(\lambda) I_{12}(2h). \tag{33}$$

Similarly, one finds

$$r(0, \lambda) = \frac{1}{\cosh h} I_{34}(-2h) R_{13}(-\lambda) R_{24}(-\lambda) \tag{34}$$

with a coupling on the quantum spaces. The regularity and unitarity properties are satisfied without modifications. The equivalent (for periodic boundary conditions) Hamiltonian is simpler:

$$\begin{aligned} \frac{d}{d\lambda} \check{r}(\lambda, \mu)|_{\lambda=\mu} &= \mathcal{P}_{13} \tilde{P}_{13}^{(3)} + \mathcal{P}_{24} \tilde{P}_{24}^{(3)} + \frac{U}{\cosh 2h} (p_{13}^{(1)} + \cos(2\mu) p_{13}^{(2)} + \sin(2\mu) p_{13}^{(3)}) \\ &\times (p_{24}^{(1)} + \cos(2\mu) p_{24}^{(2)} + \sin(2\mu) p_{24}^{(3)}). \end{aligned} \tag{35}$$

The matrix *r* shows that the Hubbard structure lies in the linear combination of two objects rather than the factors of *I*(*h*).



## 3.2. Fusion

The right-hand side of (28) shows that  $\lambda_1 - \lambda_2 = \pm\pi/2$  are possible projector points. Both values actually yield projectors with the same dimensionality. This result is peculiar to the underlying  $XX$  system, but there is otherwise no essential difference between the two projectors. For definiteness  $\rho = +\pi/2$  is considered below. Let

$$\pi_{1234}^{(1)} = \frac{1}{(x+x^{-1})(x'+(x')^{-1})} (\tilde{P}_{13}^{(1)} + \tilde{P}_{13}^{(3)}) (\tilde{P}'_{24}^{(1)} + \tilde{P}'_{24}^{(3)}) \quad (36)$$

$$\pi_{1234}^{(2)} = \mathbb{I} - \pi_{1234}^{(1)}. \quad (37)$$

The function  $g(\lambda)$  defined in (2) is constant and equal to  $(x+x^{-1})(x'+(x')^{-1})$ . The expression (36) is a decoupled product of a projector for each copy of a free-fermion system. To arrive at this result one uses the following relation which is proved by a direct calculation:

$$[\pi_{13}^{(1)} \pi_{24}^{(1)}, I_{12}(h) I_{34}(-h)] = 0 \quad \forall h \in \mathbb{C} \quad (38)$$

where  $\pi_{ij}^{(1)} = (x+x^{-1})^{-1} (\tilde{P}_{ij}^{(1)} + \tilde{P}_{ij}^{(3)})$  is the projector of one copy ( $x$  may have a different value for each copy).

The dimensions of these projectors are given by their traces. In particular, for the unprimed copy, one has  $\text{tr}(\pi^{(1)}) = n_1 n_2$  and  $\text{tr}(\pi^{(2)}) = n_1^2 + n_2^2 + n_1 n_2$ . The matrices which diagonalize one copy of both projectors are given by

$$S = \sum_{a,a'} E^{aa} \otimes E^{a'a'} + \sum_{\beta,\beta'} E^{\beta\beta} \otimes E^{\beta'\beta'} + \sum_a \sum_\beta (E^{aa} \otimes E^{\beta\beta} + x^{-1} E^{\beta\beta} \otimes E^{aa}) \\ + \sum_a \sum_\beta (E^{a\beta} \otimes E^{\beta a} - x E^{\beta a} \otimes E^{a\beta}) \quad (39)$$

$$S^{-1} = \sum_{a,a'} E^{aa} \otimes E^{a'a'} + \sum_{\beta,\beta'} E^{\beta\beta} \otimes E^{\beta'\beta'} + \frac{1}{x+x^{-1}} \sum_a \sum_\beta (x^{-1} E^{aa} \otimes E^{\beta\beta} + E^{\beta\beta} \otimes E^{aa}) \\ + \frac{1}{x+x^{-1}} \sum_a \sum_\beta (-E^{a\beta} \otimes E^{\beta a} + x E^{\beta a} \otimes E^{a\beta}). \quad (40)$$

The diagonalized projectors read

$$S^{-1} \pi^{(1)} S = \sum_a \sum_\beta E^{\beta\beta} \otimes E^{aa} \quad (41)$$

$$S^{-1} \pi^{(2)} S = \sum_{a,a'} E^{aa} \otimes E^{a'a'} + \sum_{\beta,\beta'} E^{\beta\beta} \otimes E^{\beta'\beta'} + \sum_a \sum_\beta E^{aa} \otimes E^{\beta\beta}. \quad (42)$$

To use the fusion formulae for the Hubbard models one doubles every space to unprimed and primed copies. For instance,  $R_{14}(\lambda, \mu - \rho)$  in (10) is replaced by  $R_{11'44'}(\lambda, \mu - \rho)$  or  $r_{11'44'}(\lambda, \mu - \rho)$ , obtained from (26) or (32), respectively. The calculations involved in (3) and (10) are straightforward and can be carried out using the explicit expressions (36), (37), (39) and (40). Similarly, the quadratic Hamiltonian density can eventually be obtained from the right-hand side of (14) or (15), or directly once (10) is calculated. The explicit expressions in terms of  $E$ -matrices are, however, unwieldy, complicated and unenlightening unless used for specific applications such as writing down the Hamiltonian in terms of higher-spin  $sl(2)$  generators, or for an explicit diagonalization.

It is easy to verify that the relation between the fused matrices based on  $R$  and  $r$  is a non-diagonal similarity transformation:

$$\begin{aligned}
 R_{(12)(34)}^{(i)}(\lambda_1, \lambda_2) &= S_{12}^{-1} S_{1'2'}^{-1} S_{34}^{-1} S_{3'4'}^{-1} I_{11'}(h_1) I_{22'}(-h_1) I_{33'}(h_2) I_{44'}(-h_2) \\
 &\quad \times S_{12} S_{1'2'} S_{34} S_{3'4'} r_{(12)(34)}^{(i)}(\lambda_1, \lambda_2) S_{12}^{-1} S_{1'2'}^{-1} S_{34}^{-1} S_{3'4'}^{-1} \\
 &\quad \times I_{11'}(-h_1) I_{22'}(h_1) I_{33'}(-h_2) I_{44'}(h_2) S_{12} S_{1'2'} S_{34} S_{3'4'}.
 \end{aligned} \tag{43}$$

To unravel the coupling structure of the models just obtained from fusion, we can look for Lax matrices as in (29). The following relations for (19) are easily derived:

$$R_{12}(\lambda \pm \frac{1}{2}\pi) C_1 = -C_1 R_{12}(\lambda \mp \frac{1}{2}\pi) \tag{44}$$

$$R_{12}(\lambda \pm \frac{1}{2}\pi) C_1 = -R_{12}(\pm \frac{1}{2}\pi - \lambda) C_2. \tag{45}$$

(Such relations clearly have fermionic counterparts [22].) One can then obtain the fused matrices (10) (for (26)) at some particular points:

$$\begin{aligned}
 R_{(12)(34)}^{(i)}(\lambda, 0) &= \frac{1}{\cosh^2 h} S_{12}^{-1} S_{1'2'}^{-1} I_{11'}(-h) I_{22'}(h) S_{12} S_{1'2'} R_{(12)(34)}^{(i)}(\lambda) \\
 &\quad \times R_{(1'2')(3'4')}^{(i)}(\lambda) S_{12}^{-1} S_{1'2'}^{-1} I_{11'}(h) I_{22'}(-h) S_{12} S_{1'2'}.
 \end{aligned} \tag{46}$$

and

$$\begin{aligned}
 R_{(12)(34)}^{(i)}(\lambda, \frac{1}{2}\pi) &= \frac{1}{\cosh^2 h} S_{12}^{-1} S_{1'2'}^{-1} I_{11'}(h) I_{22'}(-h) S_{12} S_{1'2'} R_{(12)(34)}^{(i)}(\lambda - \frac{1}{2}\pi) \\
 &\quad \times R_{(1'2')(3'4')}^{(i)}(\lambda - \frac{1}{2}\pi) S_{12}^{-1} S_{1'2'}^{-1} I_{11'}(-h) I_{22'}(h) S_{12} S_{1'2'}.
 \end{aligned} \tag{47}$$

These expressions correspond to the decoupled product of two copies of multi-flavour spin-1 ( $i = 2$ ), or spin-0 ( $i = 1$ ) models at their free-fermion point  $\gamma = \pi/2$ . Indeed, contrary to what happens in (29), the product of  $SIS$ 's on the left is the inverse of that on the right, and as such the  $I$ -matrices implement an innocuous gauge transformation rather than a coupling. (Another explanation of the non-coupling nature of the  $I$ 's is found in the following paragraph.) This negative result can be understood with hindsight. A simple coupling through the  $I$ -matrices would have been naive because the conjugation operator  $C$  has very special properties with respect to the  $R$ -matrices and is peculiar to the spin- $\frac{1}{2}$  representation.

The points  $\mu = 0, \pm\pi/2$ , and similarly  $\lambda = 0, \pm\pi/2$ , are then decoupling points. For a generic pair  $(\lambda, \mu)$  there is no decoupling, and (3) and (10) applied to the Hubbard models yield the multi-flavour spin- $(0 \times \frac{1}{2})$ , spin- $(1 \times \frac{1}{2})$ , spin- $(0 \times 0)$  and spin- $(1 \times 1)$  Hubbard models. The structure of their corresponding matrices has a simple interpretation. As emphasized in section 3.1, it is the special linear combination of  $RR$  and  $RCRC$  appearing in (32) which is characteristic of the Hubbard models. Looking back at (10) and expanding the product with  $R$  replaced by  $r$ , one finds a special linear combination of 16 terms, each a product of  $RR$  and  $RCRC$ . The 16 terms precisely account for all possible combinations one could have found natural to consider. Both  $R$  and  $RC$  satisfy Yang–Baxter equations [10, 21], and are therefore candidates for linear combinations. Fusion nicely retains this structure. It also shows the preponderant role played by the underlying  $sl(2)$  structure of the  $XX$  models which can be repeatedly fused to reach any spin and therefore any spin- $(s \times s')$  Hubbard model.

### 3.3. Non-locality at $q^2 = -1$

An important issue is whether a given  $R$ -matrix is regular. Within the QISM, integrability ensures the existence of a large number of commuting quantities. The Yang–Baxter equation

implies that the transfer matrices for  $N$  sites,  $\tau(\lambda, \mu) = \text{Tr}_0[R_{0N}(\lambda, \mu) \cdots R_{01}(\lambda, \mu)]$ , mutually commute at arbitrary values of either of the spectral parameters (the other one remaining fixed). The matrix  $\tau(\lambda, \mu)$  is therefore the generator of mutually commuting quantities. Such quantities are generically non-local. For periodic boundary conditions, local, commuting spin-chain Hamiltonians can, however, be defined by taking the derivatives of the logarithm of the transfer matrix at a point where the  $R$ -matrix is regular. The transfer matrix at such a point is proportional to the unit-shift operator on the chain. Its inverse, in the logarithmic derivatives, ‘cancels’ most of the operators in the numerators at the sites where no derivative has been taken. In the following cases regularity does not hold. Unitarity by itself is not enough to ensure that the transfer matrix is invertible, and each case should be considered separately.

Before turning to the fused Hubbard models, consider the spin-1 matrix which can be obtained by fusion from the  $XX$  models (19), and which appears in (46) and (47) for  $i = 2$ . The following gauge transformation turns the asymmetric  $m = 2$   $XXC$   $R$ -matrix of [28] into a symmetric matrix  $R^{(s)}$ :

$$R^{(s)}(\lambda) = (\mathbb{I} \otimes A(\lambda)) R(\lambda) (\mathbb{I} \otimes A(-\lambda)) \quad (48)$$

where  $A(\lambda) = \sum_{\alpha_1} E^{\alpha_1 \alpha_1} e^{i c_1 \lambda} + \sum_{\alpha_2} E^{\alpha_2 \alpha_2} e^{i c_2 \lambda}$  and  $c_2 - c_1 = 1$ . At  $\gamma = \pi/2$ ,

$$R^{(s)}(\lambda) = \tilde{P}^{(1)} \sin \gamma + \tilde{P}^{(2)} \sin(\gamma + \lambda) + \tilde{P}^{(3)} \sin \lambda \quad (49)$$

reduces to (19). The net effect on fusing the symmetric version, for any value of  $q$ , is to remove (before  $q^2 \rightarrow -1$ ) all factors of  $y^{\pm 1}$  and  $q^{\pm 1}$  from (38) in [28]. The symmetric spin-1 matrix is reproduced in the appendix. For  $\gamma = \pi/2$  the two simple zeros of  $f(\lambda) = \sin(\gamma + \lambda) \sin(\gamma - \lambda)$  become a double zero at  $\lambda = \pi/2$ . Setting  $\gamma = \pi/2$  in (A1) allows one to cancel out a factor of  $\sin \lambda$ . The resulting matrix satisfies the Yang–Baxter equation, is unitary but not regular at any value of  $\lambda$ . However, one can still define local Hamiltonians through a limiting procedure from generic values of  $\gamma$ . One drops the prefactor in the left-hand side of (A1), and calculates the local conserved quantities with this renormalized  $R$ -matrix:

$$H_{p+1} = (\sin \gamma \sin 2\gamma)^p \frac{d^p}{d\lambda^p} \log(\text{Tr}_0[R_{0N}(\lambda) \cdots R_{01}(\lambda)])|_{\lambda=0} \quad p \geq 0. \quad (50)$$

These commuting local Hamiltonians are finite and non-trivial as  $\gamma \rightarrow \pi/2$ . (The factor  $(\sin \gamma \sin 2\gamma)^p$  may cancel some contributions but leaves the main ones.) Thus despite the lack of a regular point it is possible to define local conserved quantities.

The function  $f(\lambda_1, \lambda_2)$  in (28) has a double zero, at  $\lambda_1 - \lambda_2 = \pm\pi/2$ , which is inherited from the  $XX$  models forming the Hubbard matrix. The special cases (46) and (47) are non-regular matrices and indicate that the fused matrix (10) ( $i = 2$ ) for the Hubbard models, is probably not regular for all values of  $\lambda = \mu$ . (For the spin- $(0 \times 0)$  case,  $i = 1$ , dimensionality considerations imply regularity, just as for the  $XX$  models.) A definite proof of non-regularity for generic values of the spectral parameters would be welcome. There is no known ‘quantum’ deformation of the generalized Hubbard models to invoke a first-order zero and take the limit. Although the absence of local quantities would be surprising, it is not clear whether they exist and how they can be calculated for all the Hubbard models corresponding to coupled spin- $(s \times s')$   $XX$  models, with  $s$  or  $s' \geq 1$ .

Finally, consider the point  $q^2 = -1$  ( $\gamma = \pm\pi/2$ ) for all the models considered in [28]. The function  $f(\lambda)$  in the unitarity equation picks up a double zero at  $\lambda = \pi/2$ . So one can expect the loss of the regularity property for all the matrices  $R_{(12)(34)}^{(i)}(\lambda)$  obtained from the  $R$ -matrices of the defining representations of the multiplicity  $A_m$  models. This was seen above explicitly for the spin-1  $A_1$  model. Further fusions will propagate this non-regularity to all

the higher representations, with the combined appearance of higher harmonics of  $\gamma$ , that is of higher roots of unity for  $q$ . Local conserved quantities should, however, still be obtainable through the limiting method described above. That the fourth roots of unity, and more generally  $n$ th roots of unity, play a specific role for the single-flavour models ( $n_i = 1, i = 1, \dots, m+1$ ) is not surprising. The representation theory of the quantum algebra  $\mathcal{U}_q(sl(m+1))$  is in one-to-one correspondence with that of the undeformed algebra when  $q$  is not a root of unity. This is not the case for roots of unity: the representation theory becomes richer and more complicated. The defining representations are, however, undeformed for all values of  $q$ .

#### 4. Conclusion

The fusion method was shown to generalize naturally to non-additive solutions of the Yang–Baxter equation. Expressions for the fused matrices, the regularity and unitarity equations and the quadratic Hamiltonians were obtained. The issue of non-simple zeros was raised and connected to a possible lack of regularity of the fused matrices. This raised the issue of the existence of a set of *local* commuting quantities. The generalized Hubbard models were then shown to allow fusion for all spin- $(s \times s')$  representations, and compact expressions were obtained for the  $R$ -matrices corresponding to mixed spin-0,  $\frac{1}{2}$ , 1 multi-flavour representations. Local Hamiltonians are believed to exist but a definite proof and a method of calculation are lacking.

The fused Hubbard models inherit the symmetries of the two coupled multi-flavour spin- $(s \times s')$  copies. These symmetries, and the fusion equations between the various transfer matrices can be used to diagonalize them through the algebraic Bethe ansatz. (The lack of regularity should not pose a problem.)

The connection, noted in section 3.1 for the Hubbard models, between the  $L$ - and  $R$ -matrices and the corresponding  $RLL$  and  $RRR$  Yang–Baxter equations could serve as a naturalness test for the choice of an  $R$ -matrix. This could be particularly relevant in the Bariev model for which more than one  $R$ -matrix is known to exist [14–16]. More generally one can ask the following question. Provided the dimensions match and given a Lax operator associated with a non-additive  $R$ -matrix, can  $L$  be obtained as a limiting case of the original  $R$  or some other one with the  $RLL$  relation satisfied? Another general test could be the existence of projector points. Fusion should also be applicable to the Bariev and Chiral Potts models.

Finally, a non-additive matrix for the Bariev model was recently obtained by twisting a quantum group  $R$ -matrix and taking a singular limit [30]. It is, however, not clear whether such a method can be made to work for the Hubbard models.

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#### Appendix. The symmetric multi-flavour spin-1 matrix

The spin-1 matrix discussed in section 3.3 is given by

$$\frac{\sin(2\gamma - \lambda)}{\sin(\lambda + \gamma)} R_{(12)(34)}^{(2)(s)}(\lambda) = + \sin(\lambda + \gamma) \sin(\lambda + 2\gamma) \sum_{a,b,c,d} E^{ab} \otimes E^{cd} \otimes E^{ba} \otimes E^{dc} \\ + \sin(\lambda + \gamma) \sin(\lambda + 2\gamma) \sum_{\alpha,\beta,\gamma,\delta} E^{\alpha\beta} \otimes E^{\gamma\delta} \otimes E^{\beta\alpha} \otimes E^{\delta\gamma}$$

$$\begin{aligned}
& + \sin(2\gamma) \sin(\lambda + \gamma) \sum_a \sum_{\alpha, \beta, \gamma} E^{\alpha\alpha} \otimes E^{\beta\gamma} \otimes E^{\alpha\alpha} \otimes E^{\gamma\beta} \\
& + \sin(2\gamma) \sin(\lambda + \gamma) \sum_{a, b, c} \sum_{\alpha} E^{ab} \otimes E^{c\alpha} \otimes E^{ba} \otimes E^{\alpha c} \\
& + \sin(2\gamma) \sin(\lambda + \gamma) \sum_{a, b, c} \sum_{\alpha} E^{ab} \otimes E^{\alpha c} \otimes E^{ba} \otimes E^{c\alpha} \\
& + \sin(2\gamma) \sin(\lambda + \gamma) \sum_a \sum_{\alpha, \beta, \gamma} E^{\alpha\alpha} \otimes E^{\beta\gamma} \otimes E^{\alpha\alpha} \otimes E^{\gamma\beta} \\
& + (\sin \gamma \sin(2\gamma) + \sin \lambda \sin(\lambda + \gamma)) \sum_{a, b} \sum_{\alpha, \beta} E^{ab} \otimes E^{\alpha\beta} \otimes E^{ba} \otimes E^{\beta\alpha} \\
& + \sin \gamma \sin(2\gamma) \sum_{a, b} \sum_{\alpha, \beta} E^{\alpha\alpha} \otimes E^{\beta\beta} \otimes E^{\alpha\alpha} \otimes E^{\beta\beta} \\
& + \sin \gamma \sin(2\gamma) \sum_{a, b} \sum_{\alpha, \beta} E^{\alpha\alpha} \otimes E^{\beta\beta} \otimes E^{\alpha\alpha} \otimes E^{\beta\beta} \\
& + 2x^{-1} \cos \gamma \sin(2\gamma) \sin \lambda \sum_{a, b} \sum_{\alpha, \beta} E^{\alpha\alpha} \otimes E^{\beta\alpha} \otimes E^{ab} \otimes E^{b\beta} \\
& + x \sin \gamma \sin \lambda \sum_{a, b} \sum_{\alpha, \beta} E^{ab} \otimes E^{\alpha\alpha} \otimes E^{b\beta} \otimes E^{\beta\alpha} \\
& + x^{-2} \sin \lambda \sin(\lambda + \gamma) \sum_{a, b, c} \sum_{\alpha} E^{ab} \otimes E^{\alpha\alpha} \otimes E^{bc} \otimes E^{ca} \\
& + x^{-2} \sin \lambda \sin(\lambda + \gamma) \sum_a \sum_{\alpha, \beta, \gamma} E^{\alpha\beta} \otimes E^{\gamma\alpha} \otimes E^{aa} \otimes E^{\beta\gamma} \\
& + x^2 \sin \lambda \sin(\lambda + \gamma) \sum_{a, b, c} \sum_{\alpha} E^{ab} \otimes E^{ca} \otimes E^{bc} \otimes E^{\alpha\alpha} \\
& + x^2 \sin \lambda \sin(\lambda + \gamma) \sum_a \sum_{\alpha, \beta, \gamma} E^{aa} \otimes E^{\alpha\beta} \otimes E^{\beta\gamma} \otimes E^{\gamma\alpha} \\
& + x^{-3} \sin \gamma \sin \lambda \sum_{a, b} \sum_{\alpha, \beta} E^{\alpha\alpha} \otimes E^{\beta\beta} \otimes E^{bb} \otimes E^{\alpha\alpha} \\
& + 2x^3 \cos \gamma \sin(2\gamma) \sin \lambda \sum_{a, b} \sum_{\alpha, \beta} E^{aa} \otimes E^{b\alpha} \otimes E^{ab} \otimes E^{\beta\beta} \\
& + x^{-4} \sin(\lambda - \gamma) \sin \lambda \sum_{a, b} \sum_{\alpha, \beta} E^{\alpha\alpha} \otimes E^{\beta\beta} \otimes E^{aa} \otimes E^{bb} \\
& + x^4 \sin(\lambda - \gamma) \sin \lambda \sum_{a, b} \sum_{\alpha, \beta} E^{aa} \otimes E^{bb} \otimes E^{\alpha\alpha} \otimes E^{\beta\beta}. \tag{A1}
\end{aligned}$$

This matrix is regular for  $\gamma \neq \frac{1}{2}\pi + k\pi$  ( $k \in \mathbb{Z}$ ), and unitary for arbitrary  $\gamma$ :

$$R_{(12)(34)}^{(2)(s)}(0) = \sin^2 \gamma \mathcal{P}_{13} \mathcal{P}_{24} \pi_{12}^{(d)} \pi_{34}^{(d)} \tag{A2}$$

$$R_{(12)(34)}^{(2)(s)}(\lambda) R_{(34)(12)}^{(2)(s)}(-\lambda) = \sin^2(\gamma + \lambda) \sin^2(\gamma - \lambda) \pi_{12}^{(d)} \pi_{34}^{(d)} \tag{A3}$$

where  $\pi^{(d)}$  is equal to the right-hand side of (42). (It is necessary to let  $x \rightarrow -x$  before using (A1) in (46) and (47).)

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